CALCULEM PRIMITIVES !

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Càlcular primitives o integrar és més complicat que derivar ja que no tenim normes pel producte i el quocient, per exemple, o la regla de la cadena. Cal practicar molt.

Anem a veure uns quants exemples de tècniques. Recordeu que les primitives són 'les inverses' de les derivades!

$\boxed{1}$ Primitives immediates (s'obtenen de llegir al revès la taula de les derivades de funcions elementals)

Primer recordeu $\int f+g = \int f + \int g$ i $\int cf = c\int f$ on $c \in \mathbb{R}$.

[amb regla cadena]

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int f^n \cdot f' = \frac{f^{n+1}}{n+1} + C$$

$$\int e^x dx = e^x + C \qquad \int e^f \cdot f' = e^f + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \qquad \int a^f f' = \frac{a^f}{\ln a} + C$$

$$\int \frac{1}{x} dx = \ln|x| + C \qquad \int \frac{f'}{f} = \ln|f| + C$$

$$\int \sin x \, dx = -\cos x + C \qquad \int (\sin f) \cdot f' = -\cos f + C$$

$$\int \cos x \, dx = \sin x + C \qquad \int \cos(f) \cdot f' = \sin f + C$$

$$\int \frac{1}{\cos^2 x} dx = \operatorname{tg} x + C \qquad \int \frac{f'}{\cos^2 f} = \operatorname{tg} f + C$$

$$\int \frac{1}{1+x^2} dx = \operatorname{arctg} x + C \qquad \int \frac{f'}{1+f^2} = \operatorname{arctg} f + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C \qquad \int \frac{f'}{\sqrt{1-f^2}} = \arcsin f + C$$

Exemple : • $\int \operatorname{tg} x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx =$

fixeu-vos $(\cos x)' = -\sin x$

$= -\ln|\cos x| + C$

$\left[\int \frac{f'}{f} = \ln|f| + C \right]$

• $\int \frac{1}{x \ln x}\, dx = \int \frac{1/x}{\ln x}\, dx = \ln|\ln x| + C$

$\left[\int \frac{f'}{f} = \ln|f| + C \right]$

• $\int \frac{x}{\sqrt{1+x^2}}\, dx = \int x(1+x^2)^{-1/2}\, dx = \frac{1}{2}\int 2x(1+x^2)^{-1/2}\, dx =$

$\left[\int f^n \cdot f' = \frac{f^{n+1}}{n+1} + C \right]$

$= \frac{1}{2}\, \frac{(1+x^2)^{-1/2+1}}{-1/2+1} + C =$

$= (1+x^2)^{1/2} + C = \sqrt{1+x^2} + C$

• $\int \frac{\sin x}{\sqrt[4]{\cos^3 x}}\, dx = \int \sin x \cdot (\cos x)^{3/4}\, dx = -\int (-\sin x)(\cos x)^{3/4} dx$

$= -\frac{(\cos x)^{3/4+1}}{3/4+1} = -\frac{4}{7} \cdot \sqrt[4]{\cos^7 x} + C$

• $\int \sin(e^{5x})\, e^{5x}\, dx = \frac{1}{5}\int \sin(e^{5x}) \cdot 5e^{5x}\, dx =$

$\int \sin f \cdot f' = -\cos f + C$

$= -\frac{1}{5}\cos(e^{5x}) + C$

- $\int \frac{x}{\sqrt{1-x^4}}\,dx = \int \frac{x}{\sqrt{1-(x^2)^2}}\,dx =$

$\int \frac{f'}{\sqrt{1-f^2}} = \arcsin f + C$

$= \frac{1}{2}\int \frac{2x}{\sqrt{1-(x^2)^2}}\,dx = \frac{1}{2}\arcsin x^2 + C$

- $\int \frac{x^3}{1+x^2}\,dx$ Aquesta no s'assembla a cap de les immediates. Fem el quocient de polinomis per simplificar-la.

$$\begin{array}{r|l} x^3 & x^2+1 \\ \cline{2-2} x^3+x & x \\ \hline -x & \end{array}$$

$\int \frac{x^3}{1+x^2}\,dx = \int x - \frac{x}{x^2+1}\,dx = \int x\,dx - \int \frac{x}{x^2+1}\,dx$

$= \int x\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1) + C.$

- $\int x\sin(x^2+1)\,dx = \frac{1}{2}\int 2x\cdot\sin(x^2+1)\,dx =$

$= -\frac{1}{2}\cos(x^2+1) + C$

$\boxed{2}$ Integració per parts: és una tècnica que ens permet 'atacar' l'integral de producte en circumstàncies especials

Recordeu $(f\cdot g)' = f'g + fg'$. Alhores si 'integrem'

$$f\cdot g = \int f'g + \int fg'$$

i per tant $\int fg' = f\cdot g - \int f'g$.

Normalment s'escriu $\int u\,dv = u\cdot v - \int v\cdot du$

Exemples

① $\int \underset{u}{\underline{x}}\,\underset{dv}{\underline{e^x}}\,dx = xe^x - \int e^x dx = xe^x - e^x + C$

$u = x \quad dv = e^x$

$du = 1 \quad v = \int e^x dx = e^x$

② $\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx =$

$( )' \quad u = x^2 \quad dv = \cos x \quad \int$

$du = 2x \quad v = \sin x$

$= x^2 \sin x - 2\int x \sin x\, dx$. Repetim el procés per calcular $\int x \sin x\, dx$.

[$\int x \sin x\, dx = -x\cos x - \int -\cos x\, dx =$

$u = x \quad dv = \sin x$

$du = 1 \quad v = -\cos x$

$= -x\cos x + \int \cos x\, dx = -x\cos x + \sin x + C$]

i canviem a l'exemple original.

$$\int x^2 \cos x \, dx = x^2 \sin x - 2\int x \sin x \, dx =$$

$$= x^2 \sin x - 2\left(-x\cos x + \sin x\right) + C =$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + C.$$

- $$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx =$$

$u = e^x \quad dv = \sin x$ $\qquad$ $u = e^x \quad dv = \cos x$

$du = e^x \quad v = -\cos x$ $\qquad$ $du = e^x \quad v = \sin x$

$$= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x \, dx\right)$$

Tenim una igualtat

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

i aïllem la primitiva que volem calcular

$$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{1}{2}\left(-e^x \cos x + e^x \sin x\right) + C$$

- $$\int \ln x \, dx = \int \ln x \cdot 1 \, dx = x \ln x - \int x \frac{1}{x} dx =$$

$u = \ln x \quad dv = 1$

$du = \frac{1}{x} \quad v = x$

$$= x \ln x - \int 1 \, dx = x \ln x - x + C.$$

Fixeu-vos que podem comprovar que el resultat és correcte fent la derivada.

$$(x\ln x - x)' = 1\cdot \ln x + x\cdot \frac{1}{x} - 1 =$$

$$= \ln x + 1 - 1 = \ln x.$$

- $$\int x \,\text{arctg}\, x \, dx = \frac{x^2}{2}\text{arctg}\, x - \int \frac{1}{2}\,\frac{x^2}{x^2+1}\,dx$$

$$\left[\begin{array}{ll} u = \text{arctg}\, x & dv = x \\ du = \frac{1}{1+x^2} & v = \frac{x^2}{2} \end{array}\right]$$

$$= \frac{x^2}{2}\text{arctg}\, x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx = \frac{x^2}{2}\text{arctg}\, x - \frac{1}{2}\int 1 - \frac{1}{x^2+1}\,dx$$

$$\begin{array}{r|l} x^2 & x^2+1 \\ \cline{2-2} -x^2-1 & 1 \\ \hline -1 & \end{array}$$

$$= \frac{x^2}{2}\text{arctg}\, x - \frac{1}{2}\left(x - \text{arctg}\, x\right) + C =$$

$$= \frac{x^2}{2}\text{arctg}\, x - \frac{1}{2}x + \frac{1}{2}\text{arctg}\, x + C$$

- $$\int \frac{\ln x}{\sqrt{x}}\,dx = \int x^{-1/2}\cdot \ln x \, dx = 2\sqrt{x}\ln x - \int \frac{1}{x}\cdot 2\sqrt{x}\,dx$$

$$\begin{array}{ll} u = \ln x & dv = x^{-1/2} \\ du = \frac{1}{x} & v = 2x^{1/2} = 2\sqrt{x} \end{array}$$

$$= 2\sqrt{x}\ln x - 2\int x^{-1/2}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

3 CANVI DE VARIABLE

El mètode del canvi de variable és aplicar la regla de la cadena al revès.

$$(f\circ g)'(x) = f'(g(x))\cdot g'(x)$$

alehores $\displaystyle\int f'(g(x))\, g'(x)\, dx = (f\circ g)(x) + C$

$$[g(x) = y \,,\; g'(x)dx = dy]$$

$$\int f'(g(x))g'(x)\,dx = \int f'(y)\,dy = f(y) + C =$$
$$= f(g(x)) + C.$$

- $\displaystyle\int \frac{\sin x}{2+\cos x}\,dx = \int \frac{-dy}{y} = -\ln|y| + C =$

$2+\cos x = y$
$-\sin x\,dx = dy \rightarrow \sin x\,dx = -dy$

$= -\ln|2+\cos x| + C$

- $\displaystyle\int (2x+1)^9 (3x-2)\,dx =$

$2x+1 = y \qquad x = \dfrac{y-1}{2}$
$2dx = dy$
$dx = \frac{1}{2}dy \qquad 3x-2 = 3\left(\dfrac{y-1}{2}\right)-2 =$
$= \frac{1}{2}(3y-7)$

$$= \int y^9 \cdot \frac{1}{2}(3y-7)\cdot\frac{1}{2}\,dy =$$
$$= \frac{1}{4}\int y^9(3y-7)\,dy = \frac{1}{4}\int 3y^{10} - 7y^9\,dy =$$

$$= \frac{1}{4}\left(3\frac{y^{11}}{11} - 7\frac{y^{10}}{10}\right) + C =$$

$$= \frac{3}{44}(2x+1)^{11} - \frac{7}{40}(2x+1)^{10} + C$$

- $$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{1}{1+y} \cdot 2y\,dy = 2\int \frac{y}{1+y}\,dy =$$

$$y = \sqrt{x} \qquad dy = \frac{1}{2\sqrt{x}}\,dx$$

$$2\sqrt{x}\,dy = dx$$

$$2y\,dy = dx$$

$$= 2\int 1 - \frac{1}{1+y}\,dy = 2\left(y - \ln|1+y|\right) + C$$

$$= 2\left(\sqrt{x} - \ln|1+\sqrt{x}|\right) + C$$

- $$\int x\sqrt{2x-1}\,dx = \int \frac{y+1}{2} \cdot \sqrt{y} \cdot \frac{dy}{2} = \frac{1}{4}\int (y+1)\,y^{1/2}\,dy =$$

$$y = 2x-1, \quad x = \frac{y+1}{2}$$

$$dy = 2dx, \quad dx = \frac{dy}{2}$$

$$= \frac{1}{4}\int y^{3/2} + y^{1/2}\,dy = \frac{1}{4}\left(\frac{y^{5/2}}{5/2} + \frac{y^{3/2}}{3/2}\right) + C =$$

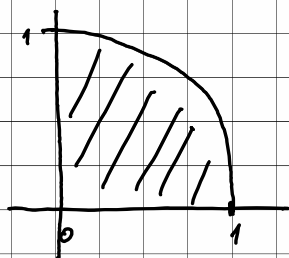
$$= \frac{1}{4} \cdot \left(\frac{2}{5}\sqrt{y^5} + \frac{2}{3}\sqrt{y^3}\right) + C =$$

$$= \frac{1}{2}\left(\frac{1}{5}\sqrt{(2x-1)^5} + \frac{1}{3}\sqrt{(2x-1)^3}\right) + C$$

Anem a fer un exemple que combina geometria i fòrmules trigonomètriques.

$$\int_0^1 \sqrt{1-x^2}\, dx$$

Mètode 1 Dibuixem la regió de la qual volem calcular l'àrea



$\frac{1}{4}$ Àrea cercle de radi 1

$$\frac{\pi}{4}$$

Mètode 2 Calcular una primitiva $\int \sqrt{1-x^2}\, dx$ i fent servir la regla de Barrow. Fem un canvi de variable, $x = \sin\theta$.

$$\int_0^1 \sqrt{1-x^2}\, dx = \int_0^{\pi/2} \sqrt{1-\sin^2\theta} \cdot \cos\theta\, d\theta = \int_0^{\pi/2} \cos^2\theta\, d\theta.$$

$x = \sin\theta$

$dx = \cos\theta\, d\theta$, fixeu-vos que si $x \in [0,1]$ aleshores $\theta \in [0, \pi/2]$

Ara bé, com calculem $\int \cos^2\theta\, d\theta$? Aquesta es resol fent servir un canvi trigonomètric que val la pena recordar

$$\cos 2x = 2\cos^2 x - 1.$$

$$\int \cos^2 x\, dx = \int \frac{1}{2}(\cos 2x + 1)\, dx =$$

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

$$= \frac{1}{2}\int \cos 2x\, dx + \frac{1}{2}\int 1\, dx = \frac{1}{4}\int 2\cos 2x\, dx + \frac{1}{2}\int 1\, dx =$$

$$= \frac{1}{4}\sin 2x + \frac{1}{2}x + C$$

Aleshores $\displaystyle\int_0^{\pi/2} \cos^2\theta\, d\theta = \left[\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right) + \frac{1}{2}\cdot\frac{\pi}{2}\right] -$

$$- \left[\frac{1}{4}\sin(2\cdot 0) + \frac{1}{2}\cdot 0\right] = \frac{\pi}{4}$$

$\boxed{4}$ Primitives racionals

Una funció racional és una funció que s'expressa com un quocient de dos polinomis

$$\frac{P(x)}{Q(x)}$$

CAS 1 : Immediates

$$\int \frac{1}{(x-a)^n}\, dx = \frac{(x-a)^{-n+1}}{-n+1} + C \qquad n \geq 2$$

$$\int \frac{1}{x-a}\, dx = \ln|x-a| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\int \frac{1/a}{\left(\frac{x}{a}\right)^2+1}\,dx = \frac{1}{a}\,\text{arctg}\left(\frac{x}{a}\right)+C$$

CAS 2 : quan grau $P(x) \geq$ grau $Q(x)$

Em aquest cas, fem la divisió de polinomis.

$$\int \frac{x^3}{x+1}\,dx = \int \underbrace{x^2-x+1}_{\text{quocient}} - \frac{\overbrace{1}^{\text{resta}}}{1+x}\,dx =$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|1+x| + C$$

$$\begin{array}{rl}
x^3 & \big|\ \ x+1 \\
-\ (x^3+x^2) & \ \ x^2-x+1 \\
\hline
-x^2 & \\
-\ (-x^2-x) & \\
\hline
+x & \\
x+1 & \\
\hline
-1 &
\end{array}$$

ó

x^3+0x^2+0x+0

$$\begin{array}{r|rrrr}
 & 1 & 0 & 0 & 0 \\
-1 & & -1 & 1 & -1 \\
\hline
 & 1 & -1 & 1 & \boxed{-1}
\end{array}$$

$x+1$

x^2-x+1 quocient

reste

CAS 3 : quan grau $P(x) <$ grau $Q(x)$, fem la descomposició de $\frac{P(x)}{Q(x)}$ en fraccions simples, és a dir, denominador $(x+a)^n$.

Si $Q(x) = (x-a_1)^{n_1} \cdots (x-a_r)^{n_r}$,

$$\frac{P(x)}{Q(x)} = \frac{?}{x-a_1} + \frac{?}{(x-a_1)^2} + \cdots + \frac{?}{(x-a_1)^{n_1}} + \frac{?}{(x-a_2)} + \cdots$$
$$\cdots + \frac{?}{(x-a_r)^{n_r}}$$

Anem a veure exemples.

- $\int \frac{x}{x^2-1}\,dx$. El denominador x^2-1 es descomposa com $x^2-1 = (x-1)(x+1)$.

Aleshores volem decomposar

$$\frac{x}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\frac{x}{x^2-1} = \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}.$$ Per tant

$$x = A(x+1) + B(x-1)$$

Avaluem l'expressió en $x=-1$, i $x=1$.

$$-1 = A\cdot 0 + B(-2), \quad B = 1/2$$
$$1 = A\cdot 2 + B\cdot 0, \quad A = 1/2$$

$$\int \frac{x}{x^2-1}\,dx = \int \frac{1/2}{x-1}\,dx + \int \frac{1/2}{x+1}\,dx = \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

- $\displaystyle\int \frac{x}{x^3+x^2-x-1}\,dx$

Anem a decomposar x^3+x^2-x-1. Comencem usant Ruffini per buscar arrels.

$$\begin{array}{r|rrrr}
 & 1 & 1 & -1 & -1 \\
1 & & 1 & 2 & 1 \\
\hline
 & 1 & 2 & 1 & \boxed{0} \\
-1 & & -1 & -1 & \\
\hline
 & 1 & 1 & \boxed{0} &
\end{array}$$

$$(x-1)(x+1)(x+1) = (x-1)(x+1)^2$$

Així $\dfrac{x}{x^3+x^2-x-1} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$$\frac{x}{x^3+x^2-x-1} = \frac{A(x+1)^2+B(x-1)(x+1)+C(x-1)}{(x-1)(x+1)^2}$$

Per tant,

$$x = A(x+1)^2+B(x-1)(x+1)+C(x-1)$$

$x=-1 \qquad -1 = C\cdot(-2), \qquad C = 1/2$

$x=1 \qquad 1 = 4A, \qquad A = 1/4$

$x=0 \qquad 0 = A-B-C, \qquad B = A-C = 1/4-1/2 = -1/4$

$$\int \frac{x}{(x-1)(x+1)^2}\,dx = \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx +$$

$$+ \frac{1}{2}\int \frac{1}{(x+1)^2}\,dx =$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\,\frac{(x+1)^{-2+1}}{-2+1} + C =$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\cdot\frac{1}{x+1} + C$$

- $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$. Fixeu-vos que $x^2-5x+6 = (x-2)(x-3)$

ja que $x^2-5x+6=0$ si $x=2$ o $x=3$.

$$\frac{1}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3)+B(x-2)}{x^2-5x+6}$$

$$1 = A(x-3) + B(x-2)$$

$x=3 \quad 1 = B$

$x=2 \quad 1 = A(-1),\ A=-1$

$$\int \frac{1}{x^2-5x+6}\,dx = -\int \frac{1}{x-2}\,dx + 1\cdot\int \frac{1}{x-3}\,dx =$$

$$= -\ln|x-2| + \ln|x-3| + C.$$